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Killing spinors and SYM on curved spaces

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Abstract: We construct two families of globally supersymmetric counterparts of standard Poincaré supersymmetric SYM theories on curved space-times admitting Killing spinors, in all dimensions less than six and eight, respectively. The former differs from the standard theory only by mass terms for the fermions and scalars and modified supersymmetry transformation rules, the latter in addition has cubic Chern-Simons like couplings for the scalar fields. We partially calculate the supersymmetry algebra of these models, finding R-symmetry extensions proportional to the curvature. We also show that generically these theories have no continuous Coulomb branch of maximally supersymmetric vacua, but that there exists a half-BPS Coulomb branch approaching the standard Coulomb branch in the Ricci-flat limit.

Keywords: Brane Dynamics in Gauge Theories, Extended Supersymmetry.

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Contents

1. Introduction

It is well known that Poincaré supersymmetric gauge theories retain a certain fraction of their supersymmetry when placed on Ricci flat manifolds M admitting covariantly constant spinors, simply by using these parallel spinors as the supersymmetry parameters. For the same reason string theory compactifications on such manifolds lead to space-time supersymmetry.

From the string or supergravity theory point of view it is almost equally natural to consider (maximally) supersymmetric compactifications of the form $M_1 \times M_2$ where this time the M_i are required to be Einstein manifolds admitting Killing spinors rather than covariantly constant spinors.

It is therefore natural to ask if super-Yang-Mills (SYM) theories retain some global supersymmetry when placed on backgrounds admitting Killing spinors. For instances, this question arises in the context of the AdS/CFT correspondence [1]–[4] when considering curved wrapped D-branes, as e.g. in [5, 6, 7]. It also ought to arise, for the same reason as in the case of branes wrapped over supersymmetric cycles of manifods admitting parallel spinors (see e.g. [8, 9]), in the context of AdS-calibrations studied in [10, 11].

Morally speaking, by virtue of the existence of Killing spinors, globally supersymmetric SYM theories should exist on such manifolds, and it should be possible to deduce their existence and properties directly, i.e. without having to pass through supergravity and the possibly arduous task of studying fluctuations around a given (perhaps not even maximally) supersymmetric background.

It appears to be almost folklore knowledge that for the four-dimensional SYM theories addition of a suitable mass term for the scalars in the vector multiplet is sufficient to ensure supersymmetry on a background with Killing spinors. However, I am not aware of any general and systematic, i.e. not tied to a particular dimension, discussion of these matters.

Here, in addition to reproducing these results for $n = 4$, we will find two families of Killing SYM theories for $n \neq 4$, both of them with the same field content as their Poincaré supersymmetric counterparts but with different actions and (generically) different supersymmetry transformation laws. From the results one can see in retrospect that the four-dimensional case (with equal masses for all the scalars, no other scalar potential terms, no mass term for the fermions) is sufficiently special to preclude a straightforward extrapolation to other dimensions.

One of these families of theories, given in (3.7), has the presumably unsurprising property of differing from the flat space theory by mass terms for the scalar fields and (unless the space-time dimension is $n = 4$) fermions. I would suspect that these theories can be readily extracted from the supergravity literature. However, even one of the simplest members of this family of theories we will find, namely the $N = 2$ theory on AdS_5, was only constructed very recently in [12], so perhaps these theories are not so well known after all.

The other family, given in (3.13), existing in all dimensions $n \leq 7$, has the more curious feature of requiring Chern-Simons-like cubic couplings of the scalar fields for supersymmetry and appears to be new.

One unexpected consequence of this is the existence of two inequivalent supersymmetric curved space counterparts of the three-dimensional $N = 4$ SYM theory on locally AdS spaces: one with with fermionic and bosonic mass terms and modified supersymmetry transformation rules, the other with the same supersymmetry transformation rules as in flat space but with a cubic interaction term for the scalars instead of a mass term.

If these Killing SYM theories are realized as world volume theories of certain curved D-branes — and the role of wrapped branes e.g. in studies of the AdS/CFT correspondence certainly suggests that they should be thought of as being equipped with a supersymmetric world volume dynamics — then certainly the fundamental properties of these theories, supersymmetric vacua, BPS configurations etc. need to be understood. Here we will just discuss one simple but intriguing aspect of these theories, namely the counterpart of what is usually called the Coulomb branch. What we will find is that the structure of the vacua with unbroken supersymmetries in these theories differs quite markedly from that in the Poincaré supersymmetric theories — e.g. in the sense that generically there is no continuous family of maximally supersymmetric vacua, i.e. all the flat directions of the potential are lifted by a contribution to the potential induced by the curvature.

This in itself may not be terribly surprising, given the known results about other quantum field theories in AdS space-times. However, it certainly calls for a reappreciation of these issues in the context of brane dynamics.

As signs are crucial when it comes to checking supersymmetry, section 2 and an appendix serve to establish the conventions and notation and to provide some background information regarding supersymmetry variations in curved backgrounds and Killing spinors. In section 3, the two classes of theories mentioned above are described, and in section 4 the supersymmetry algebra in these models is (partially) calculated. Section 5 contains some sample calculations in these models, dealing mainly with the absence of a maximally supersymmetric Coulomb branch and the existence of a half-BPS Coulomb branch.

There are a large number of open issues, e.g. a more conceptual understanding of the existence of these theories (which here have been constructed more or less by brute force), and their superalgebraic underpinning, the study of the corresponding quantum theories, spaces of vacua, BPS configurations, application to worldvolume theories of curved D-branes, etc. Work on these and related issues (the original motivation for looking at (and hence first for) these theories was part of an attempt to find a topological counterpart of the AdS/CFT correspondence) is in progress, and I will briefly come back to these issues in the concluding section 6.

JHEP11(2000)023

2. Background

2.1 SYM theories in flat space

We will consider the $N = 1$ SYM theories in $d = 2{+}1,\ 3{+}1,\ 5{+}1$ and $9{+}1$ dimensions as well as their dimensional reductions to $n \le d$ dimensions. This dimensional reduction could be along space-like directions to produce the standard Minkowski signature SYM theories, but it could also involve the time-direction to give rise to hermitean SYM actions in euclidean signature [13, 14, 15]. Thus in particular these theories include the $N = 2$ and $N = 4$ theories in $n = 3 + 1$ as well as their euclidean counterparts.

Quite generally, for all these theories the lagrangian in d or n-dimensional flat space can be written in the compact form

$$L_{\rm SYM} = -\frac{1}{2} F_{MN} F^{MN} + \bar{\Psi}\Gamma^M D_M \Psi\,. \tag{2.1}$$

Here the following conventions have been used:

- Capital indices $L, M, N, \ldots$ run from 0 to $d - 1$.
- The gauge fields A_M and Ψ only depend on the coordinates x^μ, $\mu = 0, \ldots, n-1$ or $\mu = 1, \ldots, n$ depending on whether one performs a space or time reduction. Thus A_μ is an n-dimensional gauge field and the remaining $(d-n)$ components $A_m \equiv \phi_m$ are scalar fields transforming as a vector under the manifest R-symmetry group $\mathrm{SO}(d - n)$ or $\mathrm{SO}(d - n - 1, 1)$.
- A trace is implicit in (2.1) for the interacting (non-abelian) theories, the fields transforming in the adjoint representation of the gauge group G,

  $$A_M = A^i_M T_i\,, \qquad \Psi = \Psi^i T_i\,. \tag{2.2}$$

  These Lie algebra indices will usually be suppressed in the following.
- A_M will be taken to be anti-hermitean, so that the field strength tensor is

  $$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N] \tag{2.3}$$

  (no factors of i).
- The Γ^M are d-dimensional unitary gamma matrices and satisfy

  $$\{\Gamma_M, \Gamma_N\} = \eta_{MN} \tag{2.4}$$

  with

  $$\eta_{MN} = \mathrm{diag}(-1, \underbrace{+1, \ldots, +1}_{d-1})\,. \tag{2.5}$$

- Ψ is an anticommuting spinor in d dimensions satisfying the condition

$$\begin{aligned}
d &= 2+1 : \text{Majorana}\,,\\
d &= 3+1 : \text{Majorana or Weyl}\,,\\
d &= 5+1 : \text{Weyl}\,,\\
d &= 9+1 : \text{Majorana-Weyl}\,.
\end{aligned} \tag{2.6}$$

- $\bar{\Psi}$ is the Dirac adjoint of Ψ defined by

$$\bar{\Psi} = \Psi^\dagger A_- \,, \tag{2.7}$$

where $A_- = \Gamma_0$ satisfies

$$\Gamma_M^\dagger = -A_- \Gamma_M A_-^{-1} \,. \tag{2.8}$$

- D_M is the gauge covariant derivative,

$$\begin{aligned}
D_\mu \Psi &= \partial_\mu \Psi + [A_\mu, \Psi]\,,\\
D_m \Psi &= [\phi_m, \Psi]\,.
\end{aligned} \tag{2.9}$$

With these conventions, and the rule

$$(\chi^\dagger \psi)^\dagger = -\psi^\dagger \chi \tag{2.10}$$

for anticommuting spinors χ, ψ, the above action is hermitean. Explicitly it reads

$$\begin{aligned}
L_{\mathrm{SYM}} = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} - D_\mu \phi_m D^\mu \phi^m - \frac{1}{2} [\phi_m, \phi_n][\phi^m, \phi^n] + \\
+ \bar{\Psi} \Gamma^\mu D_\mu \Psi + \bar{\Psi} \Gamma^m [\phi_m, \Psi]\,.
\end{aligned} \tag{2.11}$$

In flat space it is invariant under the supersymmetry transformations

$$\begin{aligned}
\delta A_M^i &= (\bar{\varepsilon} \Gamma_M \Psi^i - \bar{\Psi}^i \Gamma_M \varepsilon)\,,\\
\delta \Psi^i &= \Gamma^{MN} F_{MN}^i \varepsilon\,,\\
\delta \bar{\Psi}^i &= -\bar{\varepsilon} \Gamma^{MN} F_{MN}^i
\end{aligned} \tag{2.12}$$

(modulo total derivatives) when ε is a constant spinor also satisfying the condition (2.6). Here

$$\Gamma^{MN} = \frac{1}{2} \left[\Gamma^M, \Gamma^N \right]. \tag{2.13}$$

In the non-abelian case, vanishing of the quartic fermionic terms arising from the variation of the gauge field in the fermion kinetic term requires a Fierz identity to hold, which is satisfied by virtue of the conditions (2.6). The free theories are invariant under (2.12) without this requirement.

For brevity we will frequently refer to the dimensional reduction of the d-dimensional $N = 1$ theory to n dimensions as the (d, n) theory. Thus the $(10, 4)$ theory is $N = 4$ SYM in four dimensions and e.g. $(6, 5)$ refers to the five-dimensional $N = 2$ theory with one Dirac spinor (actually two symplectic Majorana spinors, hence $N = 2$) and one real scalar in addition to the five-dimensional gauge field. We will mostly consider standard space-like reductions, but following the procedure outlined in [13] one can also obtain euclidean SYM theories by performing the dimensional reduction along the time-direction. These will be discussed seperately in section 3.4.

2.2 Supersymmetry variations in curved space

Let us now consider what happens when one tries to place these theories (after the appropriate-dimensional reduction) on a curved background. To be specific, denote by (M, g) a (pseudo-)riemannian n-dimensional spin manifold with metric $g_{\mu\nu}$.

There is of course no problem with writing down the action (2.1) on M by introducing a vielbein $e^a_{\ \mu}$, a spin connection ω_μ^{ab}, etc. Just to further pin down the conventions, the spin connection part of the covariant derivative is

$$\nabla_\mu \Psi = \partial_\mu \Psi + \frac{1}{4} \Gamma_{ab} \omega_\mu^{ab} \Psi \,. \tag{2.14}$$

The real issue is whether this theory will have any supersymmetry, the point being that constant spinors ε will in general not exist on M while using non-constant supersymmetry parameters in (2.12) will lead to a non-zero variation of the action through terms depending on the derivatives of ε.

By just keeping track of the terms that depend on the (covariant) derivatives of ε, it is straightforward to compute the supersymmetry variation of the action on M and the result is (once again modulo total derivatives)

$$\delta L_{\mathrm{SYM}} = \left[(\nabla_\mu \bar{\varepsilon}) \Gamma^{NL} \Gamma^\mu \Psi + \bar{\Psi} \Gamma^\mu \Gamma^{NL} (\nabla_\mu \varepsilon) \right] F_{NL} \,. \tag{2.15}$$

The F_{NL}-terms encapsulate the curvature terms $F_{\mu\nu}$ as well as derivative terms of the scalars and scalar commutator terms. Note that the supersymmetry parameters ε are gauge singlets so that the covariant derivative $\nabla_\mu \varepsilon$ includes only the spin connection but not the gauge field. The gauge and gravitational covariant derivative will be denoted by D_M.

2.3 Killing spinor equations

The most immediate non-trivial solutions to $\delta L_{\mathrm{SYM}} = 0$ (2.15) are of course provided by parallel spinors,

$$\nabla_\mu \varepsilon = 0 \,. \tag{2.16}$$

The resulting supersymmetric theories and their euclidean/topological counterparts on Ricci-flat special holonomy manifolds are reasonably well understood (see e.g. [13, 14] and references therein) and will not be considered further in this paper.

JHEP11(2000)023

A natural generalization of a parallel spinors is a Killing spinor, i.e. a Dirac spinor η in n dimensions satisfying an equation of the form

$$\nabla_\mu \eta = \alpha \gamma_\mu \eta \,, \tag{2.17}$$

where the γ_μ are n-dimensional γ-matrices and α is some real or imaginary constant.[1] These equations have been thoroughly investigated in the supergravity and mathematics literature, at least in the case when M is compact and riemannian — see e.g. [16]–[19] and the references therein for the mathematical and Kaluza-Klein supergravity aspects, respectively. For recent work on the pseudo-riemannian case see [20].

To write this back in d-dimensional terms, it is not correct to just consider an equation like $\nabla_\mu \varepsilon = \alpha \Gamma_\mu \varepsilon$ as this would for instance be incompatible with a chirality condition on ε. Instead, we postulate the slightly more general Killing spinor equation

$$\nabla_\mu \varepsilon = \alpha \Gamma_\mu \Gamma \varepsilon \,, \tag{2.18}$$

where ε denotes the d-dimensional (chiral, Majorana, ...) spinor and where Γ could be an arbitrary element of the Clifford algebra generated by the Γ^M. In fact we will be more specific than that and consider the case in which Γ is a monomial constructed from the "internal" gamma matrices Γ^m, i.e. a completely anti-symmetrized product of $0 \leq p \leq d-n$ gamma matrices. When it is necessary to indicate the degree p, we will write $\Gamma^{[p]}$ instead of Γ. Then one in particular has $(\Gamma)^2 = \pm \mathbb{I}$. Generalizations of this are certainly possible but will not be explored here.

This equation now preserves chirality when p is odd, and so it can also be used in the theories arising upon dimensional reduction of the chiral N=1 theories. Moreover, the freedom in the choice of Γ may allow one to find different supersymmetric theories for a given field content (on manifolds satisfying either the same or different integrability conditions of the Killing spinor equation). We will see examples of this below.

Finally, this generalized Killing spinor equation, when written out in n-dimensional terms, will always reduce to the standard Killing spinor equation of the type (2.17) for (appropriate linear combinations of) n-dimensional Dirac spinors,[2] and therefore the standard existence criteria for ordinary Killing spinors can be applied to (2.18).

[1] Actually, while in the mathematics literature the name Killing spinor is usually reserved for spinors satisfying (2.17), in the supergravity literature any equation of the form $\nabla_\mu \eta = M_\mu(x)\eta$ arising from setting to zero the gravitino variation in a bosonic background is called a Killing spinor equation. Here $M_\mu(x)$ is typically made up from contractions of supergravity antisymmetric tensor background fields with gamma matrices, hence the explicit x-dependence. Here we have no such background fields, and thus we are left with (2.17).

[2] or perhaps to some simple variant thereof when n is even,

$$\nabla_\mu \eta = i\alpha \gamma_\mu \gamma^{(n+1)} \eta$$

(here $\gamma^{(n+1)}$ is the chirality operator). This equation can be mapped to the standard equation (2.17) by passing to the unitarily equivalent representation $\tilde{\gamma}_\mu = i\gamma_\mu \gamma^{(n+1)}$.

2.4 Integrability conditions

The (first) integrability condition arising from the Killing spinor equation (2.18) is, taking commutators and recalling (2.14),

$$\frac{1}{4}\Gamma_{ab}\Omega^{ab}_{\ \ \mu\nu}\varepsilon = \alpha^2[\Gamma_\nu\Gamma, \Gamma_\mu\Gamma]\varepsilon\,, \tag{2.19}$$

where $\Omega^{ab}_{\ \ \mu\nu}$ denotes the curvature tensor of the spin connection $\omega^{ab}_{\ \ \mu}$. Upon contraction with Γ^ν this leads to

$$R_{\mu\nu}\Gamma^\nu\varepsilon = -2\alpha^2 g_{\mu\nu}\left[(n-2)\Gamma\Gamma^\nu\Gamma + \Gamma^\nu\Gamma^\lambda\Gamma\Gamma_\lambda\Gamma\right]\varepsilon\,. \tag{2.20}$$

For $\Gamma = \Gamma^{[p]}$ "internal" in the sense described before, so that $\Gamma^{[p]}$ commutes (anti-commutes) with all the Γ^μ if p is even (odd), one finds

$$R_{\mu\nu}\Gamma^\nu\varepsilon = -4\alpha^2(-1)^p(\Gamma^{[p]})^2(n-1)g_{\mu\nu}\Gamma^\nu\varepsilon\,. \tag{2.21}$$

In the riemannian case, an equation of the form $A_{\mu\nu}\Gamma^\nu\varepsilon = 0$ implies $A_{\mu\nu} = 0$. This can be seen by multiplying by $A^\mu_{\ \lambda}\Gamma^\lambda$. Thus in this case (2.20) implies that

$$R_{\mu\nu} = -4\alpha^2(-1)^p(\Gamma^{[p]})^2(n-1)g_{\mu\nu}\,, \tag{2.22}$$

and hence that (M, g) is an Einstein manifold. In particular, for $\Gamma = \mathbb{I}$ or, equivalently, for the ordinary Killing spinor equation (2.17), one obtains

$$R_{\mu\nu} = -4\alpha^2(n-1)g_{\mu\nu}\,. \tag{2.23}$$

Thus Killing spinors (2.17) for imaginary α (referred to as *real* Killing spinors in the mathematics literature) lead to positive curvature, and spinors with real α (imaginary Killing spinors) lead to negative curvature.

This unfortunate clash in terminology is due to the fact that typically in the mathematics literature the conventions for Clifford algebras are such that $\{\Gamma_\mu, \Gamma_\nu\} = -2g_{\mu\nu}$, the opposite of the convention used here. Perhaps a more invariant and informative terminology would have been to call a Killing spinor positive or negative according to whether the integrability condition leads to positive or negative curvature, and we will adopt this terminology from now on.

In general, the sign of the curvature depends on α, p and on $(\Gamma^{[p]})^2 = \pm\mathbb{I}$. For the chiral $N = 1$ theories and their descendants, p has to be odd in order for the Killing spinor equation to be compatible with the chirality of ε.

The integrability condition (2.22) is not sufficient for the existence of Killing spinors (not every Einstein manifold admits Killing spinors) but fortunately an analysis of the higher integrability conditions can be side-stepped by relating Killing spinors on M to parallel spinors on another Ricci flat manifold and hence establishing existence of Killing spinors directly — see [17] for positive Killing spinors and [21, 16] for negative Killing spinors.

In the pseudo-riemannian case, (2.22) is neither necessary nor sufficient. An argument like the above only leads to the conclusion that for each value of μ the vector $V_{(\mu)}$ with components $V_{(\mu)}^{\;\nu} = A_\mu^{\;\nu}$ is null, with the additional constraint $g^{\mu\nu}A_{\mu\nu} = 0$. In the case of parallel spinors, the resulting *Ricci-null* lorentzian manifolds which are not Ricci flat were recently investigated in detail in [22] (see also [23]). By the same token, one might suspect that there are non-Einstein lorentzian manifolds admitting Killing spinors. There are indeed such examples for negative pseudo-riemannian Killing spinors whereas a pseudo-riemannian manifold admitting a positive Killing spinor is necessarily Einstein [20]. Nevertheless, in the following we will simply assume that (2.22) holds. In this way we will certainly miss some solutions (in the negative curvature case), but as a first orientation this is good enough.

2.5 The Supersymmetry variation for killing spinors

In order to plug (2.18) into the formula (2.15) for δL_{SYM}, one first needs an expression for $\nabla_\mu \bar{\varepsilon}$. By using the fact that

$$(\Gamma^{[p]})^\dagger = \eta_p A_- \Gamma^{[p]} A_-^{-1}\,, \tag{2.24}$$

where

$$\eta_p = (-1)^{\binom{p+1}{2}}\,, \tag{2.25}$$

one obtains

$$\nabla_\mu \bar{\varepsilon} = -\eta_p \alpha^* \bar{\varepsilon}\Gamma\Gamma_\mu\,. \tag{2.26}$$

Thus

$$\begin{aligned} \delta L_{\mathrm{SYM}} &= \Big[-\eta_p \alpha^* \bar{\varepsilon}\Gamma\Gamma_\mu \Gamma^{NL}\Gamma^\mu \Psi + \alpha \bar{\Psi}\Gamma^\mu\Gamma^{NL}\Gamma_\mu\Gamma\varepsilon\Big] F_{NL} \\ &= 2\,\mathrm{Re}(\bar{\Psi}\Gamma^\mu\Gamma^{NL}\Gamma_\mu\Gamma\varepsilon)) F_{NL}\,. \end{aligned} \tag{2.27}$$

Splitting the gamma matrices $\{\Gamma^M\} = \{\Gamma^\mu, \Gamma^m\}$ and using the standard identities

$$\begin{aligned} \Gamma_\mu \Gamma^{\nu\lambda}\Gamma^\mu &= (n-4)\Gamma^{\nu\lambda}\,, \\ \Gamma_\mu \Gamma^{\nu m}\Gamma^\mu &= (n-2)\Gamma^{\nu m}\,, \\ \Gamma_\mu \Gamma^{lm}\Gamma^\mu &= n\Gamma^{lm}\,, \end{aligned} \tag{2.28}$$

one can evaluate this to find

$$\begin{aligned} \delta L_{\mathrm{SYM}} = (n-4) &\left[-\eta_p \alpha^* \bar{\varepsilon}\Gamma\Gamma^{\nu\lambda}\Psi + \alpha\bar{\Psi}\Gamma^{\nu\lambda}\Gamma\varepsilon\right] F_{\nu\lambda} + \\ &+ 2(n-2)\left[-\eta_p \alpha^* \bar{\varepsilon}\Gamma\Gamma^{\nu m}\Psi + \alpha\bar{\Psi}\Gamma^{\nu m}\Gamma\varepsilon\right] D_\nu\phi_m + \\ &+ n\left[-\eta_p \alpha^* \bar{\varepsilon}\Gamma\Gamma^{lm}\Psi + \alpha\bar{\Psi}\Gamma^{lm}\Gamma\varepsilon\right][\phi_l, \phi_m]\,. \end{aligned} \tag{2.29}$$

Barring numerical coincidences, it is clear that this expression can only vanish when the expression in brackets vanishes all by itself, i.e. when

$$\mathrm{Re}(\alpha\bar{\Psi}\Gamma^{NL}\Gamma\varepsilon) = 0 \qquad \forall\, N, L\,. \tag{2.30}$$

This is only possible if $\alpha = 0$ so that one is dealing with ordinary parallel spinors (and hence Ricci flat geometries in the euclidean case and a few more possiblities for lorentzian signature).

However, there is one numerical coincidence which occurs when $d = n = 4$. In that case only the first line of (2.29) is present, but multiplied by $n - 4 = 0$. Thus e.g. for any solution to the ordinary Killing spinor equations (2.17) the $N = 1$ theory in $d = 3 + 1$ has a supersymmetry. The relevant gamma matrix identity shows that this is due to the fact that SYM theory is a theory of (non-abelian) one-forms, and one might want to speculate about an analogous result for (non-abelian?) two-form theories in $d = 5 + 1 \cdots$

3. Supersymmetric SYM theories in curved space

On the basis of these preliminaries we can now write down two families of Dirac-Yang-Mills theories in curved space which are globally supersymmetric courtesy of the existence of solutions to a suitable Killing spinor equation. These theories generically differ from the simple SYM action L_{SYM} by mass terms for both the scalars and the fermions and by a modified supersymmetry transformation rule for Ψ. In addition, one class of these theories curiously has Chern-Simons-like cubic couplings for the scalar fields. Both of these families of theories turn out (*a priori* for no good reason) to be particularly simple in four dimenions, $n = 4$, and we will start with that particular case.

3.1 Theories for $n = 4$

Let L_{SYM} be the $(d, 4)$ lagrangian, that is the dimensional reduction of the d-dimensional theory to $4 = 3 + 1$ dimensions, suitably covariantized, of course. Consider the lagrangian

$$\begin{aligned} L &= L_{\text{SYM}} \mp 8\alpha^2 \sum_{m=1}^{d-n} \phi_m^2 \\ &= -\frac{1}{2} F_{MN} F^{MN} + \bar{\Psi}\Gamma^M D_M \Psi \mp 8\alpha^2 \sum_{m=1}^{d-n} \phi_m^2 \,. \end{aligned} \tag{3.1}$$

This action is invariant under the supersymmetry transformations (suppressing the Lie algebra labels on the fields)

$$\begin{aligned} \delta A_M &= (\bar{\varepsilon}\Gamma_M \Psi - \bar{\Psi}\Gamma_M \varepsilon) \,, \\ \delta\Psi &= \Gamma^{MN} \varepsilon F_{MN} - 4\alpha \sum_{m=1}^{d-n} \phi_m \Gamma^m \Gamma \varepsilon \,, \end{aligned} \tag{3.2}$$

provided that ε satisfies the Killing spinor equation

$$\nabla_\mu \varepsilon = \alpha \Gamma_\mu \Gamma \varepsilon \,, \tag{3.3}$$

where Γ is any odd, internal matrix with $\Gamma^2 = \pm \mathbb{I}$. Here α has to be real for the $d = 4$ and $d = 10$ Majorana(-Weyl) theories, but can be either real or imaginary for the $d = 4, 6$ Weyl theories.

Indeed it is easy to see that due to the modification of the Ψ-transformation the standard variation of L_{SYM} given in (2.29) is exactly cancelled. But now one picks up terms linear in the scalar fields ϕ_m from the Killing spinor equation, namely when the derivative D_M in the fermionic kinetic term hits ε in the second term of $\delta\Psi$. This gives a term proportional to α^2,

$$\delta L_{\mathrm{SYM}} = \pm 16 \alpha^2 [\bar{\varepsilon} \Gamma^m \Psi - \bar{\Psi} \Gamma^m \varepsilon] \phi_m \,, \tag{3.4}$$

which is of course cancelled precisely by the variation of the mass term for the scalars.

Remarks.

(1) We have just recovered the folklore statement that addition of mass terms for the scalars is sufficient to render four-dimensional SYM theories supersymmetric in a background admitting Killing spinors, provided that also the supersymmetry transformation rules of the fermions are changed appropriately.

(2) In particular, the mass term is precisely the conformally invariant mass term arising in the conformally invariant wave operator

$$\Box - \frac{1}{4} \frac{n-2}{n-1} R \,, \tag{3.5}$$

where R is the scalar curvature

$$R = \pm 4 \alpha^2 n(n-1) \,. \tag{3.6}$$

(3) Note the striking similarity of the supersymmetry transformations with those of the special (i.e. superconformal) supersymmetry transformations as given e.g. in [24, 25, 26].

(4) Similar linear terms in the transformations of the fermions also appear e.g. in the Wess-Zumino model in a curved background [27] and are a rather generic feature of AdS supersymmetry — for a recent review see [28].

(5) Looking at the integrability conditions deduced before we learn that in particular the counterpart of the four-dimensional $N = 2$ theory can be supersymmetric on Einstein manifolds of either positive or negative curvature admitting solutions of the Killing spinor equation, depending on whether α is chosen to be real or imaginary.

(6) Likewise, the $N = 4$ theory can be supersymmetric in both cases, depending on whether one chooses $\Gamma = \Gamma^{[1]}$ or $\Gamma = \Gamma^{[3]}$, with α real in both cases.

(7) Even though the choice of Γ singles out one (or three) "internal" directions, all the scalars have the same mass. This is a feature that will not persist in $n \neq 4$.

(8) There is no mass term for the fermions. Once again, this is a feature peculiar to the $n = 4$ theories.

(9) Finally, it may be possible to construct this theory as a rigid limit of conformal supergravity in four dimensions.[3]

3.2 Family A: theories for $n \leq 5$ with $\Gamma = \Gamma^{[1]}$

We will now consider the case where Γ is just a single internal gamma matrix which we will call Γ^1. In particular, $(\Gamma)^2 = +\mathbb{I}$. Now consider the following action:

$$L = L_{\text{SYM}} - 4\alpha^2 \left[(n-2) \sum_{m=1}^{d-n} \phi_m^2 + (n-4)\phi_1^2 \right] - (n-4)\alpha\bar{\Psi}\Gamma^1\Psi \,. \qquad (3.7)$$

As it stands this action makes sense for the $(d = 4,\ n < 4)$, $(d = 6,\ n < 6)$, and the fermionic mass term is hermitean provided that α is imaginary (cf. appendix A) and this rules out the $(d = 10,\ n \neq 4)$ theories. We could also allow $(d = 10,\ n = 4)$ and α real, but in that case the action reduces to the one discussed above.

This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta A_M &= (\bar{\varepsilon}\Gamma_M\Psi - \bar{\Psi}\Gamma_M\varepsilon)\,, \\ \delta\Psi &= \Gamma^{MN}\varepsilon F_{MN} - 4\alpha \left[\sum_{m=1}^{d-n} \phi_m\Gamma^m\Gamma^1\varepsilon + (n-4)\phi_1\varepsilon \right] \end{aligned} \qquad (3.8)$$

provided that ε satisfies the Killing spinor equation

$$\nabla_\mu\varepsilon = \alpha\Gamma_\mu\Gamma^1\varepsilon \,. \qquad (3.9)$$

Remarks.

(1) We now have mass terms both for the scalars and the fermions. The masses depend only on the space-time dimension n, not on the parent dimension d.

(2) The mass of ϕ_1 differs from that of the $\phi_{m\neq1}$, but neither is the conformally invariant value unless $n = 2$ when $d - n - 1$ of the scalars are massless.

(3) The integrability conditions tell us that these theories can only exist on Einstein manifolds of negative curvature — in particular locally AdS space-times.

[3] I thank Ergin Sezgin for this suggestion.

JHEP11(2000)023

(4) The $(6,5)$-theory on AdS_5 has been constructed recently by Shuster [12] in terms of symplectic Majorana spinors. It can be checked that, when these are reassembled into a Dirac spinor, his action and supersymmetries agree precisely with those given above when one sets $d = 6$, $n = 5$.

(5) The R-symmetry of the action has been reduced from $\mathrm{SO}(d-n)$ (which is the manifest R-symmetry group of the Poincaré supersymmetric theory) to $\mathrm{SO}(d-n-1)$.

The simplest of these theories is the $(4,3)$ theory, i.e. the $N = 2$ theory in $n = 3$. It differs from L_{SYM} only by the fermionic mass term, and the supersymmetry transformation rules are the standard ones, i.e. we have

$$(d=4,\ n=3) \qquad \begin{aligned} L &= L_{\mathrm{SYM}} + \alpha\bar{\Psi}\Gamma^1\Psi\,, \\ \delta\Psi &= \Gamma^{MN}\varepsilon F_{MN}\,. \end{aligned} \tag{3.10}$$

This theory is supersymmetric almost by inspection. For $n = 3$, the two first lines of (2.29) enter with opposite signs and the third line is absent. As Γ_1 anticommutes with the Γ^μ but commutes with the $\Gamma^{\mu\nu}$, this is cancelled by the variation of the above fermionic mass term.

Let us also write down explictly the $(6,3)$-theory. It is given by

$$(d=6,\ n=3) \qquad \begin{aligned} L &= L_{\mathrm{SYM}} - 4\alpha^2(\phi_2^2+\phi_3^2) + \alpha\bar{\Psi}\Gamma^1\Psi\,, \\ \delta\Psi &= \Gamma^{MN}\varepsilon F_{MN} - 4\alpha(\phi_2\Gamma^2+\phi_3\Gamma^3)\Gamma^1\varepsilon\,. \end{aligned} \tag{3.11}$$

3.3 Family B: theories for $n \leq 7$ with $\Gamma = \Gamma^{[3]}$

If we want to use $\Gamma = \Gamma^{[3]}$ and still insist on this being an element of the "internal" Clifford algebra, we obviously need $n \leq d-3$. Let us choose

$$\Gamma = \Gamma^{123} = \frac{1}{3!}\varepsilon_{abc}\Gamma^{abc}\,, \tag{3.12}$$

so that $(\Gamma)^2 = -\mathbb{I}$. Consider the action

$$\begin{aligned} L = L_{\mathrm{SYM}} &+ 4\alpha^2\left[(n-2)\sum_{m=1}^{d-n}\phi_m^2 + (n-4)\sum_{a=1}^{3}\phi_a^2\right] - \\ &- \frac{(n-4)\alpha}{3!}\varepsilon_{abc}\left[\bar{\Psi}\Gamma^{abc}\Psi - 8\phi^a[\phi^b,\phi^c]\right]. \end{aligned} \tag{3.13}$$

Hermiticity of the mass term requires $\alpha \in \mathbb{R}$.

This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta A_M &= (\bar{\varepsilon}\Gamma_M\Psi - \bar{\Psi}\Gamma_M\varepsilon)\,, \\ \delta\Psi &= \Gamma^{MN}\varepsilon F_{MN} - 4\alpha\left[\sum_{m=1}^{d-n}\phi_m\Gamma^m + (n-4)\sum_{a=1}^{3}\phi_a\Gamma^a\right]\Gamma^{123}\varepsilon \end{aligned} \tag{3.14}$$

provided that ε satisfies the Killing spinor equation

$$\nabla_\mu\varepsilon = \alpha\Gamma_\mu\Gamma^{123}\varepsilon\,. \tag{3.15}$$

JHEP11(2000)023

Remarks.

(1) The most striking property of this action is perhaps the appearance of the cubic term for the scalar fields. It looks like the dimensional reduction of a standard Chern-Simons term living in the three internal directions singled out by Γ^{123}.

(2) It is certainly suggestive of a supergravity origin of this term, but it would be desirable to find a pure gauge theory explanation for it as well.

(3) Such terms can appear in the completely T-duality invariant D-brane world-volume actions discussed by Myers in [29], where they arise due to the coupling to non-trivial background antisymmetric tensor fields.

(4) Some such term also appears in the off-shell *rheonomic* formulation of $N = 1$ $d = 10$ SYM in flat space — see [30, (II.9.41)]. The relation to the appeareance of such a term in the on-shell space-time action here is not clear (to the author) but may be worth understanding.

(5) The integrability conditions once again lead to negative curvature because even though α is now real, one also has $\Gamma^2 = -\mathbb{I}$.

Apart from the $(d = 10,\ n = 4)$ theory already discussed above, for which there are neither fermionic mass terms nor Chern-Simons like couplings, the simplest theory is once again the three-dimensional $(6, 3)$-theory with lagrangian and supersymmetry transformation

$$\begin{aligned}(d = 6,\ n = 3) L &= L_{\text{SYM}} + \alpha \left(\bar{\Psi}\Gamma^{123}\Psi - 8\phi^1[\phi^2, \phi^3]\right), \\ \delta\Psi &= \Gamma^{MN}\varepsilon F_{MN}\,. \end{aligned} \tag{3.16}$$

It is straightforward to check directly in this case that the action is supersymmetric: upon performing the supersymmetry variation, the terms involving $F_{\mu\nu}$ and $D_\mu\phi_n$ arising from the variation of L_{SYM} and the fermionic mass term cancel whereas those involving the commutator $[\phi_m, \phi_n]$ add up. The latter are then precisely cancelled by the variation of the cubic scalar term.

Note that we now have two obviously inequivalent curved space versions of the $(6, 3)$-theory, i.e. of what in standard parlance be called the three-dimensional $N = 4$ SYM theory ($N = 4$ because in $2 + 1$ dimensions spinors are two-component real: $\text{so}(2, 1) \sim \text{sl}(2, \mathbb{R})$), one of them with a standard mass term for two of the three scalars (3.11), the other one instead with a Chern-Simons like term (3.16). Is there any interesting (duality?) relationship between these theories?

3.4 Euclidean supersymmetric SYM theories in curved space

Euclidean (or better perhaps: riemannian) versions of the theories described above may be of interest for a variety of reasons, e.g. for D-brane instantons, within the euclidean approach to the AdS/CFT correspondence, and in connection with Hull's E-branes [31] and an eye towards cohomological versions of these theories.

As explained in [13] (see also [14, 15]), a convenient way to obtain manifestly hermitean euclidean SYM theories is by time-like dimensional reduction of any one of the standard Minkowskian SYM theories to a lagrangian L_{ESYM}.

This construction naturally explains the features one has in the past come to expect of euclidean supersymmetric theories, such as non-compact R-symmetry groups (namely the internal roation group which is now the Lorentz group $\mathrm{SO}(d-n-1,1)$) and kinetic terms with the "wrong" sign (namely the time-compoent of the gauge field, now a scalar from the point of view of the euclidean space-time).

These theories then also automatically make sense on riemannian manifolds and retain some fraction of their supersymmetry when this manifold admits parallel spinors. In this way one obtains cohomological theories on special holonomy manifolds with many beautiful features, studied for example from this point of view in [13, 14].

Now let us, in analogy with what we did before, discuss the extension of these euclidean SYM theories to supersymmetric theories on riemannian manifolds admitting Killing spinors. Let us start with the $n=4$ theories of section 3.1. It is readily seen that the theory as it stands is supersymmetric also for the euclidean theory provided that the mass term is chosen to be $\sim \eta^{mn}\phi_m\phi_n$, i.e.

$$L = L_{\mathrm{ESYM}} \mp 8\alpha^2\eta^{mn}\phi_m\phi_n\,, \tag{3.17}$$

for any choice of (internal, odd) Γ. In particular, Γ could be chosen to be equal to (or include) Γ^0. The interesting thing about this is that according to (2.22) this changes the sign of the integrability condition. Whereas for $\Gamma=\Gamma^1$, say, the sign of the curvature is the sign of α^2, for $\Gamma=\Gamma^0$ it is minus the sign of α^2.

This may not be of great consequence in the present example since, as we saw before, we could anyhow obtain both signs by either choosing α to be real or imaginary (for the $(6,4)$ theory) or by choosing $\Gamma=\Gamma^{[1]}$ or $\Gamma=\Gamma^{[3]}$ (for the $(10,4)$ theory) — the integrability condition only depends on the square of $\alpha\Gamma$.

Moreover, for $n=4$, but only for $n=4$, there is practically no dependence of the action on Γ (apart from the sign of the mass term) so that we do not get any essentially new theories in this way. But we will see below that in the other theories the freedom to choose Γ to include or not to include Γ^0 gives us an added flexibility not present in the pseudo-riemannian theories.

More or less the same modifications as above are required for the Family A theories of section 3.2. Provided that we define the mass terms as above and reintroduce the dependence of the sign of the mass term on $\Gamma^2=\pm\mathbb{I}$, as above, we obtain a supersymmetric lagrangian. Thus essentially the only two different possibilities are

$$\begin{aligned}
\Gamma=\Gamma^1 \qquad & L = L_{\mathrm{ESYM}} - 4\alpha^2\left[(n-2)\eta^{mn}\phi_m\phi_n + (n-4)\phi_1^2\right] - (n-4)\alpha\bar{\Psi}\Gamma^1\Psi\,,\\
\Gamma=\Gamma^0 \qquad & L = L_{\mathrm{ESYM}} + 4\alpha^2\left[(n-2)\eta^{mn}\phi_m\phi_n - (n-4)\phi_0^2\right] - (n-4)\alpha\bar{\Psi}\Gamma^0\Psi\,.
\end{aligned} \tag{3.18}$$

We know that α has to be imaginary for hermiticity of the fermionic mass term (this condition is the same for $\bar{\Psi}\Gamma^1\Psi$ and $\bar{\Psi}\Gamma^0\Psi$), and previously this forced the manifold to have negative curvature. However, now we actually gain something by being able to choose $\Gamma = \Gamma^0$ or $\Gamma = \Gamma^1$ (of course, in order to have this choice one needs $n \leq d-2$). Namely, the euclidean theory now has a supersymmetric version for negative curvature ($\Gamma = \Gamma^1$) and another supersymmetric version for positive curvature, when $\Gamma = \Gamma^0$.

Mutatis mutandis one can draw the same conclusions for the theories of section 3.3. The mass terms require the same treatment as before, and the only novelty is the Chern-Simons-like cubic coupling for the scalar field. If one chooses $\Gamma = \Gamma^{123}$, no further explanation is required. On the other hand, if one chooses, say, $\Gamma = \Gamma^{012}$, then one obviously has to take into account the minus sign implicit in using $\phi^a = \eta^{ab}\phi_b$. Thus explicitly the Chern-Simons term reads

$$\frac{1}{3!}\epsilon_{abc}\phi^a[\phi^b,\phi^c] = -\phi_0[\phi_1,\phi_2]\,. \tag{3.19}$$

The only thing worth noting here is perhaps that, unlike an ordinary Chern-Simons term, which contains a first order time derivative, this algebraic term remains real in euclidean signature. The payoff from using Γ^{012} is that this theory exists on manifolds of positive curvature (admitting solutions of the corresponding Killing spinor equation, of course). Thus we have essentially the following two lagrangians:

$$\begin{aligned}
\Gamma = \Gamma^{123} \qquad L &= L_{\mathrm{ESYM}} + 4\alpha^2\Big[(n-2)\eta^{mn}\phi_m\phi_n + (n-4)\delta^{ab}\phi_a\phi_b\Big] - \\
&\quad - (n-4)\alpha\left[\bar{\Psi}\Gamma_{123}\Psi - 8\phi_1[\phi_2,\phi_3]\right], \\
\Gamma = \Gamma^{012} \qquad L &= L_{\mathrm{ESYM}} - 4\alpha^2\Big[(n-2)\eta^{mn}\phi_m\phi_n + (n-4)\eta^{ab}\phi_a\phi_b\Big] + \\
&\quad + (n-4)\alpha\left[\bar{\Psi}\Gamma_{012}\Psi - 8\phi_0[\phi_1,\phi_2]\right].
\end{aligned} \tag{3.20}$$

We see that whereas in the pseudo-riemannian case we had the freedom to choose either positive or negative curvature space-times only for $n = 4$, in the riemannian case the theories have this property for all n, subject to the restrictions $n \leq d-2$ for the A theories and $n \leq d-4$ for the B theories. In $d-1$ (respectively $d-3$) dimensions, there is no choice, Γ is dictated by whether one makes a purely spacelike or a (space-)time-reduction.

4. Aspects of the supersymmetry algebra

In order to gain some insight into the structure of the theories introduced above, and to attempt to understand them from the (A)dS superalgebra point of view, in the following we will now (partially) calculate the supersymmetry algebras in these models.

JHEP11(2000)023

4.1 The superalgebra for family A

Using (3.2), it is straightforward to calculate the commutator of two supersymmetry transformations δ_i, associated with Killing spinors $\varepsilon_1, \varepsilon_2$ satisfying $\nabla_\mu \varepsilon_i = \alpha \Gamma_\mu \Gamma_1 \varepsilon_i$, acting on the bosonic fields A_μ and ϕ_m. One finds

$$\begin{aligned} \frac{1}{4}[\delta_1, \delta_2] A_\mu &= V^N F_{N\mu} + (n-3)(\alpha + \alpha^*) V_\mu \phi_1 + (\alpha - \alpha^*) V_{\mu i 1} \phi^i \,, \\ \frac{1}{4}[\delta_1, \delta_2] \phi_1 &= V^N F_{N1} + (n-3)(\alpha + \alpha^*) V_1 \phi_1 - (\alpha + \alpha^*) V_i \phi^i \,, \\ \frac{1}{4}[\delta_1, \delta_2] \phi_j &= V^N F_{Nj} + (n-3)(\alpha + \alpha^*) V_j \phi_1 + (\alpha + \alpha^*) V_1 \phi_j + (\alpha - \alpha^*) V_{ij1} \phi^i \,. \end{aligned} \tag{4.1}$$

Here we have introduced the notation

$$\begin{aligned} V_M &= \bar{\varepsilon}_1 \Gamma_M \varepsilon_2 - \bar{\varepsilon}_2 \Gamma_M \varepsilon_1 \,, \\ V_{MNP} &= \bar{\varepsilon}_1 \Gamma_{MNP} \varepsilon_2 - \bar{\varepsilon}_2 \Gamma_{MNP} \varepsilon_1 \,. \end{aligned} \tag{4.2}$$

Ordinarily, i.e. in Poincaré supersymmetry, one would just find the first term on the right-hand side. Acting on the scalar fields, this is just the Lie derivative (diffeomorphism) with resepect to V^μ plus a field dependent gauge transformation,

$$\begin{aligned} V^N F_{Nm} &= L_V \phi_m + \delta_V \phi_m \,, \\ \delta_V \phi_m &= [V^N A_N, \phi_m] \,. \end{aligned} \tag{4.3}$$

Here and in the following it should be understood that the V in the Lie derivative refers only to the space-time components V^μ whereas all components V^M enter in δ_V.

The same is true for the gauge field *provided that V_m is constant*, as is the case for parallel spinors. In that case, one has

$$\begin{aligned} \nabla_\mu V_m = 0 \Longrightarrow V^N F_{N\mu} &= L_V A_\mu + \delta_V A_\mu \\ \delta_V A_\mu &= -D_\mu (V^N A_N) \,. \end{aligned} \tag{4.4}$$

However, when the V_m are not constant, then one has instaed

$$\nabla_\mu V_m \neq 0 \Longrightarrow V^N F_{N\mu} = L_V A_\mu + \delta_V A_\mu + (\nabla_\mu V_m) \phi^m \,. \tag{4.5}$$

In order to understand how the right hand side of the supersymmetry algebra, including also all the other new terms, nevertheless manages to be an invariance of the lagrangian in this case, we need to know some properties of the objects V_M and V_{MNP}. The following identities are easily verified:

$$\begin{aligned} \nabla_\mu V_1 &= -(\alpha + \alpha^*) V_\mu \,, \\ \nabla_\mu V_i &= (\alpha - \alpha^*) V_{1 i \mu} \,, \\ \nabla_\mu V_{ij1} &= -(\alpha + \alpha^*) V_{\mu i j} \,, \\ \nabla_\mu V_\nu &= (\alpha + \alpha^*) g_{\mu\nu} V_1 + (\alpha - \alpha^*) V_{1 \nu \mu} \,. \end{aligned} \tag{4.6}$$

In particular, therefore, V_μ is a Killing vector if $\alpha^* = -\alpha$, and a conformal Killing vector (and a gradient vector) if $\alpha^* = \alpha$. In the former case, V_1 and the antisymmetric matrices V_{1ij} are constant, whereas the other space-time scalars V_i are not (and vice-versa for α real). Moreover, note that the above equations imply that for α real the function $V_1^2 + V_\mu V^\mu$ is constant.

Using these results, we learn that the commutator of supersymmetry transformations on the gauge field can be written as

$$\frac{1}{4}[\delta_1, \delta_2]A_\mu = L_V A_\mu + \delta_V A_\mu - (n-4)(\nabla_\mu V_1)\phi_1 \,. \tag{4.7}$$

But since V_1 is constant for imaginary α and real α is only allowed for $n = 4$, we see that in all cases the last term actually disappears and the commutator takes the standard form

$$\frac{1}{4}[\delta_1, \delta_2]A_\mu = L_V A_\mu + \delta_V A_\mu \,. \tag{4.8}$$

If α is imaginary, then the commutator on the scalars takes the form

$$\begin{aligned} \alpha^* = -\alpha \Longrightarrow \frac{1}{4}[\delta_1, \delta_2]\phi_1 &= L_V\phi_1 + \delta_V\phi_1 \\ \frac{1}{4}[\delta_1, \delta_2]\phi_j &= L_V\phi_j + \delta_V\phi_j + 2\alpha V_{ij1}\phi^i \,. \end{aligned} \tag{4.9}$$

We see that in addition to diffeomorphisms (along a Killing vector) and gauge transformations, the algebra now also includes a rotation of the scalar fields by the constant matrix V_{ij1} — this is (a subgroup of) the R-symmetry algebra of the theory and, combined with an appropriate transformation of the fermions, a separate invariance of the lagrangian. The appearance of the R-symmetry algebra in the commutator of supersymmetries is of course a well-known feature of AdS superalgebras [32] (for a recent review of AdS supersymmetry see [28]) which we have recovered here somewhat experimentally. Note that this extra rotation only appears for $n \leq d-3$. In particular, it is absent for $n = 4$.

The case $\alpha^* = \alpha$ (and thus $n = 4$) is a bit more complicated, but this should not be too surprising as now V_μ is only a conformal Killing vector,

$$L_V g_{\mu\nu} = 4\alpha V_1 g_{\mu\nu} \,, \tag{4.10}$$

and additional scale transformations of the scalars and fermions are required to produce an invariance of the lagrangian density in that case. Recall that precisely when $n = 4$ the scalar field action is conformally invariant so that this is feasible in principle.

The transformation on the gauge field is, as we have noted above, the standard one, which is fine since the Yang-Mills lagrangian is conformally invariant precisely when $n = 4$. The scalars now transform as

$$\begin{aligned} \alpha^* = +\alpha \Longrightarrow \frac{1}{4}[\delta_1, \delta_2]\phi_1 &= (L_V + 2\alpha V_1)\phi_1 + \delta_V\phi_1 + \Delta_V\phi_1 \\ \frac{1}{4}[\delta_1, \delta_2]\phi_j &= (L_V + 2\alpha V_1)\phi_j + \delta_V\phi_j + \Delta_V\phi_j \,. \end{aligned} \tag{4.11}$$

Here the modified Lie derivative $L_V + 2\alpha V_1$ reflects the non-trivial conformal weight of the scalar fields, and

$$\begin{aligned} \Delta_V \phi_1 &= -2\alpha V^i \phi_i \,, \\ \Delta_V \phi_j &= 2\alpha V_j \phi_1 \end{aligned} \tag{4.12}$$

is a particular global (the V_i are constant in this case) infinitesimal SO$(d-4)$ rotation of the $(d-4)$ scalar fields. This is only non trivial for $d = 6$ and for $d = 10$. In the former case we find an SO(2) rotation parametrized by $2\alpha V_2$, namely

$$\begin{aligned} \Delta_V \phi_1 &= -2\alpha V_2 \phi_2 \,, \\ \Delta_V \phi_2 &= 2\alpha V_2 \phi_1 \,. \end{aligned} \tag{4.13}$$

Note that in this case (α real) the bosonic generators of the algebra are conformal Killing vector fields that are also gradient vector fields (this is an extremely restrictive condition but solutions exist e.g. in de Sitter space). As a consequence, since the Lie bracket of two gradient vector fields is always zero, and also commutators of the modified operators $L_V + 2\alpha V_1$ can be seen to vanish, the bosonic part of the algebra engendered in this way is abelian, a situation apparently not covered by Nahm's classification [32].

4.2 The complete superalgebra for $n = 4$

Of course, to complete this discussion we should also calculate the commutator of two supersymmetry transformations on the fermions. At this point, because now Fierz identities are required, the discussion becomes somewhat dimension-dependent and we will only do this for $n = 4$ which in many respects is the most interesting case to consider anyway.

For the $(6, 4)$-theory, the supersymmetry variation of the spinor Ψ is

$$\begin{aligned} \delta\Psi &= \Gamma^{MN} \varepsilon F_{MN} - 4\alpha \phi_m \Gamma^m \Gamma^1 \varepsilon \\ &= \Gamma^{MN} \varepsilon F_{MN} - 4\alpha(\phi_1 + \phi_2 \Gamma^{21})\varepsilon \,. \end{aligned} \tag{4.14}$$

It follows that

$$\begin{aligned} \delta_1 \delta_2 \Psi &= 2D_M(\bar\varepsilon_1 \Gamma_N \Psi - \bar\Psi \Gamma_N \varepsilon_1)\Gamma^{MN}\varepsilon_2 - \\ &\quad - 4\alpha(\bar\varepsilon_1 \Gamma_m \Psi - \bar\Psi \Gamma_m \varepsilon_1)\Gamma^m \Gamma^1 \varepsilon_2 \,. \end{aligned} \tag{4.15}$$

By the usual Fierz identity for SYM theories, the terms involving ε_1 and ε_2 will drop out after taking commutators and we drop them henceforth. From the other terms we find, using the Killing spinor equation

$$\nabla_\mu \bar\varepsilon = \alpha^* \bar\varepsilon \Gamma_1 \Gamma_\mu \,, \tag{4.16}$$

that

$$\delta_1\delta_2\Psi = 2\alpha^*\bar{\varepsilon}\Gamma_1\Gamma_\mu\Gamma_N\Psi\Gamma^{\mu N}\varepsilon_2 + 2\bar{\varepsilon}_1\Gamma_N D_M\Psi\Gamma^{MN}\varepsilon_2 - 4\alpha\bar{\varepsilon}_1\Gamma_m\Psi\Gamma^m\Gamma^1\varepsilon_2\,. \tag{4.17}$$

Taking commutators and using the Fierz rearrangement formula for Weyl spinors Ψ_k of the same chirality in d dimensions,

$$\bar{\Psi}_1 M\Psi_2\ \bar{\Psi}_3 N\Psi_4 = -2^{-d/2}\sum_{p=0}^{n/2} c_p\bar{\Psi}_1\Gamma^{[p]}\Psi_4\ \bar{\Psi}_3 N\Gamma_{[p]}M\Psi_2\,, \tag{4.18}$$

(here a sum over the antisymmetrized products of p gamma matrices is understood) with

$$\begin{aligned} c_p &= (-1)^{\binom{p}{2}}\frac{2}{p!}\,, \qquad p<\frac{n}{2}\,, \\ c_{n/2} &= (-1)^{\binom{n/2}{2}}\frac{1}{(n/2)!}\,, \end{aligned} \tag{4.19}$$

one obtains

$$\begin{aligned} [\delta_1,\delta_2]\Psi &= -\frac{1}{8}2\sum_p c_p\left(\bar{\varepsilon}_1\Gamma^{[p]}\varepsilon_2 - \bar{\varepsilon}_2\Gamma^{[p]}\varepsilon_1\right)\times \\ &\quad\times\left[\Gamma^{MN}\Gamma_{[p]}\Gamma_N D_M\Psi + \alpha^*\Gamma^{\mu N}\Gamma_{[p]}\Gamma_1\Gamma_\mu\Gamma_N\Psi - 2\alpha\Gamma^m\Gamma^1\Gamma_{[p]}\Gamma_m\Psi\right]. \end{aligned} \tag{4.20}$$

Now evidently only $p=1$ and $p=3$ contribute to the sum (this follows e.g. from the discussion leading to (A.10)), giving rise to terms involving the vectors V^M and antisymmetric tensors V^{MNP} encountered before. Upon using the equation of motion $\Gamma^M D_M\Psi = 0$, the first term will just give the standard contribution proportional to

$$V^M D_M\Psi = V^\mu\nabla_\mu\Psi + [V^N A_N,\Psi]\,. \tag{4.21}$$

This has almost the right structure to be of the form diffeomorphism plus gauge transformation we encountered for the bosonic fields. However, the (covariant) derivative on the spinor alone is not part of the invariance of the action, i.e. the fermioic kinetic term is not invariant under

$$\delta\Psi = V^\mu\nabla_\mu\Psi \tag{4.22}$$

even if V is Killing. Rather, for (conformal) Killing vectors the Lie derivatives on the bosonic fields have to be supplemented by the Lie derivative of the spinor field defined by

$$L_V\Psi = V^\mu\nabla_\mu\Psi + \frac{1}{4}\nabla_\mu V_\nu\Gamma^{\mu\nu}\Psi\,. \tag{4.23}$$

Let us note here that in the present case the second term only contributes when V is a Killing vector (α imaginary), because V is not only a conformal Killing vector but also a gradient vector when α is real. This additional contribution to the covariant derivative arises from the $p = 3$ contributions to the second and third terms in (4.20) in the form

$$\nabla_\mu V_\nu \Gamma^{\mu\nu} = (\alpha^* - \alpha) V_{\mu\nu 1} \Gamma^{\mu\nu} \,. \tag{4.24}$$

The other $p = 1$ contributions give rise to new terms in the supersymmetry algebra. After an altogether not particularly fascinating calculation one finds

$$\frac{1}{4}[\delta_1, \delta_2]\Psi = L_V \Psi + \delta_V \Psi + \frac{1}{2}(\alpha + 5\alpha^*) V_1 \Psi + \frac{1}{2}(\alpha + \alpha^*) V_i \Gamma^{i1} \Psi \,, \tag{4.25}$$

where $i \neq 1$. Now let us take a look at this for α real and imaginary, respectively. For α imaginary, the complete commutator algebra reads

$$\begin{aligned}
\frac{1}{4}[\delta_1, \delta_2] A_\mu &= L_V A_\mu + \delta_V A_\mu \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_m &= L_V \phi_m + \delta_V \phi_m \,, \\
\frac{1}{4}[\delta_1, \delta_2] \Psi &= L_V \Psi + \delta_V \Psi - 2\alpha V_1 \Psi \,.
\end{aligned} \tag{4.26}$$

Thus the only term we find in addition to the Lie derivative along a Killing vector and a gauge transformation is a constant (V_1 is constant) phase rotation (α is imaginary) of the spinor. The latter is of course an invariance of the Dirac action — in fact it is the diagonal U(1) subgroup of the SU(2) R-symmetry of the six-dimensional Weyl action. It is nevertheless interesting that this additional phase transformation appears in the commutator algebra for non-zero curvature. Its appearance in the $(6, 5)$ theory has been noted in [12].

For α real, as we had seen before, already the algebra on the bosonic fields is more complicated. In this case we have

$$\begin{aligned}
\frac{1}{4}[\delta_1, \delta_2] A_\mu &= L_V A_\mu + \delta_V A_\mu \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_1 &= (L_V + 2\alpha V_1) \phi_1 + \delta_V \phi_1 + \Delta_V \phi_1 \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_j &= (L_V + 2\alpha V_1) \phi_j + \delta_V \phi_j + \Delta_V \phi_j \,, \\
\frac{1}{4}[\delta_1, \delta_2] \Psi &= (L_V + 3\alpha V_1) \Psi + \delta_V \Psi + \alpha V_i \Gamma^{i1} \Psi \,.
\end{aligned} \tag{4.27}$$

Once again we see the modified Lie derivative on the spinor field (the factor of 3 reflecting the familiar conformal weight 3/2 of a spinor field). We also see the constant R-symmetry transformation

$$\Delta_V \Psi = \alpha V_i \Gamma^{i1} \Psi \tag{4.28}$$

accompanying the rotation $\Delta_V \phi_m$ of the scalar fields. It is now straightforward to check that this indeed constitutes an invariance of the action, as it should.

4.3 The superalgebra for family B

We will now calculate the action of the commutator of two supersymmetry transformations on the bosonic fields for the family of lagrangians (3.13) with supersymmetry transformation (3.14). Instead of V_M and V_{MNP}, this algebra will now contain in addition to the vector V_M the rank five anti-symmetric tensor

$$V_{MNPQR} = \bar{\varepsilon}_1 \Gamma_{MNPQR} \varepsilon_2 - \bar{\varepsilon}_2 \Gamma_{MNPQR} \varepsilon_1 \,. \tag{4.29}$$

A straightforward calculation gives

$$\begin{aligned}
\frac{1}{4}[\delta_1, \delta_2] A_\mu &= V^N F_{N\mu} - 2\alpha V_{123i\mu} \phi^i \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_a &= V^N F_{Na} + 2(n-3)\alpha \varepsilon_{abc} \phi^b V^c \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_i &= V^N F_{Ni} + 2\alpha V_{123ij} \phi^j \,.
\end{aligned} \tag{4.30}$$

To interpret this, we proceed as in the analysis of (4.1). First of all we note the following properties:

$$\begin{aligned}
\nabla_\mu V_a &= 0 \,, \\
\nabla_\mu V_i &= 2\alpha V_{123i\mu} \,, \\
\nabla_\mu V_\nu &= -2\alpha V_{123\mu\nu} \,, \\
\nabla_\mu V_{123ij} &= 0 \,, \\
\nabla_\mu V_{123} &= 0 \,.
\end{aligned} \tag{4.31}$$

This shows that V_μ is a Killing vector and that the coefficients of the scalar field rotations are constants. There is an SO(3) rotation acting on the three scalar fields ϕ_a and an SO$(d-n-3)$ rotation on the remaining scalars ϕ_i. The last relation, which we will only need later, shows that V_{123} is a constant, an imaginary constant to be precise.

Moreover, the second relation allows us to write, as before,

$$\begin{aligned}
\frac{1}{4}[\delta_1, \delta_2] A_\mu &= V^\nu F_{\nu\mu} - V^m D_\mu \phi_m - (\nabla_\mu V_i)\phi^i \\
&= L_V A_\mu + \delta_V A_\mu \,,
\end{aligned} \tag{4.32}$$

so that all in all we have

$$\begin{aligned}
\frac{1}{4}[\delta_1, \delta_2] A_\mu &= L_V A_\mu + \delta_V A_\mu \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_a &= L_V \phi_a + \delta_V \phi_a + \Delta_V \phi_a \,, \\
\frac{1}{4}[\delta_1, \delta_2] \phi_i &= L_V \phi_i + \delta_V \phi_i + \Delta_V \phi_i \,,
\end{aligned} \tag{4.33}$$

JHEP11(2000)023

where

$$\begin{aligned}\Delta_V \phi_a &= 2(n-3)\alpha\varepsilon_{abc}\phi^b V^c\,,\\ \Delta_V \phi_i &= 2\alpha V_{123ij}\phi^j\,.\end{aligned}\tag{4.34}$$

Let us consider two special cases of this. The first is the $(6,3)$ theory. In this case evidently the commutator algebra is just the standard algebra, in agreement with the fact that the supersymmetry transformations themselves are just the standard ones — see (3.16). However, we will see below that in spite of this the commutator algebra acting on Ψ is different.

The second is the $(10,4)$ theory, i.e. the curved space counterpart of $N=4$ SYM theory in four dimensions. In this case we see that the supersymmetry algebra exhibits an SO(3) × SO(3) R-symmetry. I.e. from the point of view of the Poincaré supersymmetric theory the presence of curvature has broken the R-symmetry down from SO(6) to SO(4) $\sim$ SO(3) × SO(3). This is in perfect agreement with what a look at the AdS superalgebras would lead one to conclude. The relevant superalgebra is now not the superconformal USp($N=4|4$) with its SU(4) R-symmetry but the AdS superalgebra

$$\mathrm{OSp}(N=4|4)\supset O(3,2)\times \mathrm{SO}(4)\,.\tag{4.35}$$

It is rather pleasing to note that in the present context this reduction of the R-symmetry group can be traced back directly to the fact that the relevant Killing spinor equation involves the object Γ_{123}. This itself came from the requirement of having a hermitean fermionic mass term for spinors that started off as ten-dimensional Majorana-Weyl spinors.

4.4 The complete superalgebra for $n=3$

We have seen above that in the $(6,3)$ theory the supersymmetry transformations (3.16) and the supersymmetry commutator algebra on the bosonic fields (4.34) are just the usual ones, and one might suspect that this essentially forces the commutator algebra on the fermionic fields to be the standard one as well. However, this is not necessarily the case.

First of all we know that the standard derivative term in the algebra has to be promoted to the spinorial Lie derivative (4.23) along a Killing vector field.

Secondly, in calculating $[\delta_1,\delta_2]\Psi$ one encounters derivatives of the spinor parameters and in this way the fact that the ε_i are Killing spinors rather than parallel spinors feeds itself into the algebra.

Thirdly, in calculating this algebra one makes use of the Ψ-equations of motion. A look at the action (3.16) reveals that these are

$$\Gamma^M D_M\Psi+\alpha\Gamma_{123}\Psi=0\,,\tag{4.36}$$

and therefore no longer describe a massless spinor.

And indeed one finds a new term in the commutator algebra even in this case, where such a term is not required by similar terms in the bosonic algebra. Starting from

$$\frac{1}{4}[\delta_1, \delta_2]\Psi = -\frac{1}{16}\sum_p c_p V^{[p]} \left[\Gamma^{MN}\Gamma_{[p]}\Gamma_N D_M \Psi - \alpha\Gamma^{\mu N}\Gamma_{[p]}\Gamma_{123}\Gamma_\mu\Gamma_N\Psi\right] , \qquad (4.37)$$

one finds that the first term contributes

$$\text{1st term} = \alpha V^L D_L\Psi + \frac{3}{8}\alpha V^N\Gamma_N\Gamma_{123}\Psi + \frac{1}{96}\alpha V^{[3]}\Gamma_{[3]}\Gamma_{123}\Psi , \qquad (4.38)$$

which does not look particularly encouraging. However, the second term gives rise to

$$\begin{aligned}\text{2nd term} = {} & \frac{5}{8}\alpha V^\mu\Gamma_\mu\Gamma_{123}\Psi - \frac{3}{8}\alpha V^a\Gamma_a\Gamma_{123}\Psi - \\ & - \frac{3}{96}\alpha V^{[3]}\Gamma_{[3]}\Gamma_{123}\Psi - \frac{1}{96}\alpha V^{[3]}\epsilon^{abc}\Gamma_a\Gamma^\mu\Gamma_{[3]}\Gamma_\mu\Gamma_{bc}\Psi . \qquad (4.39)\end{aligned}$$

The "mixed" three-index terms, i.e. those involving $V^{\mu\nu a}$ and $V^{\mu ab}$ cancel, while the other two, those involving $V^{\mu\nu\lambda}$ and V^{abc}, add up and (using the chirality of Ψ) give rise to a single term proportional to $V_{123}\Psi$. The net result is then

$$\frac{1}{4}[\delta_1, \delta_2]\Psi = V^L D_L\Psi + \alpha V^\mu\Gamma_\mu\Gamma_{123}\Psi - 2\alpha V_{123}\Psi . \qquad (4.40)$$

The second term is the missing contribution for the spinorial Lie derivative (4.23) as can be seen by using (4.31) and calculating

$$\begin{aligned}\frac{1}{4}\nabla_\mu V_\nu\Gamma^{\mu\nu}\Psi &= -\frac{1}{2}\alpha V_{123\mu\nu}\Gamma^{\mu\nu}\Psi \\ &= \alpha V^\mu\Gamma_\mu\Gamma_{123}\Psi , \qquad (4.41)\end{aligned}$$

where the second equality follows from the chirality of Ψ. Thus finally we have

$$\frac{1}{4}[\delta_1, \delta_2]\Psi = L_V\Psi + \delta_V\Psi - 2\alpha V_{123}\Psi , \qquad (4.42)$$

and only the last term requires some conmment. As we have seen in (4.31), V_{123} is constant and, in fact, $(V_{123})^\dagger = -V_{123}$, so that V_{123} is an imaginary constant. But then the lagrangian (3.16) is obviously invariant under this phase rotation of the fermions. Once again, as in (4.26), we find that the Killing spinor supersymmetry algebra includes this phase rotation for $\alpha \neq 0$, i.e. for curved spaces.

5. The coulomb branch: some sample calculations

5.1 Family A: absence of a maximally supersymmetric coulomb branch

Recall that the standard Poincaré supersymmetric SYM theory has the lagrangian (2.11)

$$\begin{aligned}L_{\text{SYM}} = {} & -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu\phi_m D^\mu\phi^m - \frac{1}{2}[\phi_m, \phi_n][\phi^m, \phi^n] + \\ & + \bar{\Psi}\Gamma^\mu D_\mu\Psi + \bar{\Psi}\Gamma^m[\phi_m, \Psi] \qquad (5.1)\end{aligned}$$

and the fermionic supersymmetry transformation (2.12)

$$\delta\Psi = \Gamma^{\mu\nu}\varepsilon F_{\mu\nu} + 2\Gamma^{\mu m}\varepsilon D_\mu\phi_m + \Gamma^{mn}\varepsilon[\phi_m, \phi_n]\,. \tag{5.2}$$

The quartic potential has flat directions for mutually commuting scalar fields. Thus there is a family of vacua parametrized by the constant expectation values of the scalar fields taking values in the Cartan subalgebra of the gauge group (modulo the action of the Weyl group). The supersymmetry transformations of the fermions are identically zero in such a background without any condition on ε, and thus these configurations parametrize a family of maximally supersymmetric vacua of the SYM theory, the Coulomb branch.

We will now look for analogues of these solutions in the Killing SYM theories we have discussed above, and we will see that typically (because of the modified supersymmetry transformations and scalar potentials) there are no configurations which have all of the above properties, but that there are half-supersymmetric configurations which reduce to the above in the limit of vanishing curvature.

We begin by exploring the presence of a maximally supersymmetric purely scalar field configuration in the theories of section 3. We will first consider the family A theories for $d = 6$ and $d = 10$. The supersymmetry variation of the fermions in a purely scalar background becomes

$$\delta\Psi = 2\Gamma^{\mu n}\varepsilon\partial_\mu\phi_n + \Gamma^{mn}\varepsilon[\phi_m, \phi_n] - 4\alpha\left[\sum_{m=1}^{d-n}\Gamma^m\Gamma^1\varepsilon\phi_m + (n-4)\varepsilon\phi_1\right]. \tag{5.3}$$

It is clear almost by inspection that, unless $n = 3$ and without any further constraints on ε beyond the chirality constraint dictated by d-dimensional supersymmetry, vanishing of $\delta\Psi$ implies vanishing of all the ϕ_m because of the terms in $\delta\Psi$ linear in the ϕ_m.

Indeed, first of all vanishing of the terms proportional to $\Gamma^{\mu m}$ requires $\partial_\mu\phi_m = 0$. The term linear in ϕ_1, proportional to the identity matrix acting on ε has to vanish sepreately, so one has ϕ_1=0. The coefficient of Γ^{k1}, $k \neq 1$, is proportional to $[\phi_k, \phi_1] - 2\alpha\phi_k = -2\alpha\phi_k$, and therefore also all the other scalar fields have to vanish, ϕ_k=0.

An exception occurs for $n = 3$, as ϕ_1 does then not appear in the term in brackets proportional to α and can therefore be chosen to be constant but otherwise unconstrained. By gauge invariance, this constant can be chosen to lie in the Cartan subalgebra of the gauge group.

Thus for $n \neq 3$ there are no non trivial maximally supersymmetric purely scalar configurations (switching on any scalar vev breaks at least some fraction of the supersymmetry), while for $n = 3$ there is (for $G = \mathrm{SU}(2)$) a one-dimensional Coulomb “twig”.

If these gauge theories can be shown to arise as worldvolume theories of branes, this should have implications for the possibility (or lack thereof) to move them apart, and thus also for the question of existence of marginal bound states among these branes.

5.2 Family B: existence of a discrete family of maximally supersymmetric scalar field configurations

For the family B theories, with their supersymmetry variation

$$\delta\Psi = \Gamma^{MN}\varepsilon F_{MN} - 4\alpha \left[\sum_{m=1}^{d-n} \phi_m \Gamma^m + (n-4)\sum_{a=1}^{3}\phi_a\Gamma^a\right]\Gamma^{123}\varepsilon\,, \tag{5.4}$$

the situation is somewhat different.

In particular, as we had seen in (3.16), α disappears altogether from the supersymmetry transformation rules for $(d=6, n=3)$. In that particular case, we therefore find the "normal" Coulomb branch parametrized by the three constant commuting scalars. These solutions are also the only maximally supersymmetric critical points of the scalar cubic plus quartic potential.

For the reductions of the $d = 10$ theories to $n \leq 7$ dimensions the situation is the following. We once again set the gauge fields to zero. Then imposing $\delta\Psi = 0$ forces the scalars to be constants. The terms linear in the ϕ_k, $k \neq 1, 2, 3$ are proportional to $\Gamma^{k123}\varepsilon$ and have to vanish seperately. Thus $\phi_k = 0$. For the remaining scalar fields ϕ_a, by looking at the coefficients of $\Gamma^{ab}\varepsilon$ we find the condition

$$[\phi_a, \phi_b] = 2\alpha(n-3)\epsilon_{abc}\phi_c\,. \tag{5.5}$$

Up to an irrelevant scaling, this amounts to a homomorphism of the Lie algebra of SU(2) into that of the gauge group G and hence there are maximally supersymmetric vacua for each conjugacy class of such homomorphisms.

It can also be checked directly that this gives a critical point of the potential (with $\phi_m = 0$ for $m \neq 1, 2, 3$)

$$V(\phi) = -\frac{1}{2}\operatorname{Tr}[\phi_a, \phi_b]^2 + 8\alpha^2(n-3)\operatorname{Tr}\phi_a^2 + \frac{4}{3}\alpha(n-4)\epsilon_{abc}\phi_a[\phi_b, \phi_c]\,. \tag{5.6}$$

This is reminiscent of the analysis by Vafa and Witten [33] of the vacua of the mass-perturbed $N = 4$ SYM theory: in that case the cubic superpotential of the $N = 4$ theory (in $N = 1$ language) is perturbed by quadratic mass terms, and the equation for the critical points is equivalent to (5.5).[4] Here we find this solution even in the presence of an additional quartic term in the potential.

We see that for these theories there are indeed maximally supersymmetric vacua, but that their structure is rather different from that of the standard Coulomb branch. Instead of a continous we have a discrete family of vacua with unbroken supersymmetry, and this is reflected in the absence of flat directions in the scalar potential for $n \neq 3$.

[4]For a recent discussion of these theories in the context of the AdS/CFT correspondence see [34].

5.3 Existence of a half-BPS Coulomb branch for AdS space-times

In order to study configurations preserving some fraction of the supersymmetry, we need to know what kind of additional conditions can be imposed on an n-dimensional Killing spinor. Clearly a chirality condition (which is the natural condition for constant or parallel spinors) is incompatible with the Killing spinor equation

$$\nabla_\mu \eta = \alpha \gamma_\mu \eta \,. \tag{5.7}$$

Fortunately, very compact and explicit expressions are known [35] for Killing spinors on AdS space-times, and these results will enable us to find half-supersymmetric scalar field configurations.

We begin by quickly reviewing the results obtained in [35]. The AdS_n metric takes a particularly simple form in horospheric (or the closely related Poincaré) coordinates, in which one has

$$ds^2 = dr^2 + \mathrm{e}^{2r/\ell} \eta_{ij} dx^i dx^j \,. \tag{5.8}$$

The scalar curvature of this metric is

$$R = -\frac{1}{\ell^2} n(n-1) \,, \tag{5.9}$$

which identifies ℓ as the curvature radius of the space-time, related to our constant α by $|\alpha| = 1/2\ell$. The spinorial covariant derivative in these coordinates is

$$\begin{aligned} \nabla_r \eta &= \partial_r \eta \,, \\ \nabla_k \eta &= \partial_k \eta + \frac{1}{2\ell} \gamma_k \gamma_r \eta \,. \end{aligned} \tag{5.10}$$

Hence the Killing spinor equation

$$\nabla_\mu \eta = \frac{1}{2\ell} \gamma_\mu \eta \tag{5.11}$$

can be written as the pair of equations

$$\begin{aligned} \partial_r \eta &= \frac{1}{2\ell} \gamma_r \eta \,, \\ \partial_k \eta &= \frac{1}{2\ell} \gamma_k (1 - \gamma_r) \eta \,. \end{aligned} \tag{5.12}$$

Clearly, if $\gamma_r \eta = \eta$, the solutions are

$$\eta^+ = \mathrm{e}^{\frac{r}{2\ell}} \eta_0^+ \,, \tag{5.13}$$

where η_0^+ is an arbitrary constant spinor satisfying

$$\gamma_r \eta_0^+ = \eta_0^+ \,. \tag{5.14}$$

JHEP11(2000)023

These are the Killing spinors we will consider in the following. The general solution is

$$\eta = \mathrm{e}^{\frac{r}{2\ell}\gamma_r} \left(1 + \frac{1}{2\ell} x^k \gamma_{\underline{k}}(1-\gamma_r)\right) \eta_0 \,, \tag{5.15}$$

where η_0 is now an arbitrary constant spinor and $\gamma_{\underline{k}}$ refers to an orthonormal basis. This shows that AdS has the maximal number of linearly independent Killing spinors, i.e. is maximally supersymmetric in the supergravity sense.

Armed with these solutions to the Killing spinor equations, we can now reconsider the issue of supersymmetric purely scalar field configurations. For concreteness we consider the Family A $(6, n)$ theories for $n = 4$ and $n = 5$.

For AdS_5 we choose gamma-matrices $\gamma_k,\ k = 0, 1, 2, 3$ satisfying

$$\{\gamma_k, \gamma_l\} = \mathrm{e}^{2r/\ell} \eta_{kl} \tag{5.16}$$

and $\gamma_r = \gamma^{(5)}$. A convenient basis for the $d = 6$ Clifford algebra is then

$$\begin{aligned} \Gamma_k &= \sigma_1 \otimes \gamma_k \qquad k = 0, \ldots, n-2 = 3 \,, \\ \Gamma_r &= \sigma_1 \otimes \gamma^{(5)} \,, \\ \Gamma_5 &= \sigma_2 \otimes \mathbb{I} \,, \end{aligned} \tag{5.17}$$

where we have now, for sanity's sake, called the internal gamma matrix appearing in the Killing spinor equation

$$\nabla_\mu \varepsilon = \alpha \Gamma_\mu \Gamma_5 \varepsilon \,, \tag{5.18}$$

Γ_5 instead of Γ_1. For $n = 4$ we will choose a dimensional reduction along the x^3-direction so that now $\{\gamma_\mu\} = \{\gamma_k, \gamma^{(5)}\}$ with $k = 0, 1, 2$.

For ε a six-dimensional Weyl spinor, $\varepsilon^T = (\eta^T, 0)$ the Killing spinor equation reduces to

$$\nabla_\mu \eta = i\alpha \gamma_\mu \eta \tag{5.19}$$

so we have the identification

$$i\alpha = \frac{1}{2\ell} \,. \tag{5.20}$$

Therefore the AdS Killing spinor equation becomes

$$\begin{aligned} \nabla_k \eta &= \frac{1}{2\ell} \gamma_k \eta \,, \\ \nabla_r \eta &= \frac{1}{2\ell} \gamma^{(5)} \eta \,, \end{aligned} \tag{5.21}$$

so that indeed $\gamma_r = \gamma^{(5)}$ and the condition $\gamma_r \eta = \eta$ translates into a standard chirality condition in the four-dimensional sense.

We begin with the $n = 5$ theory, denote the single scalar field simply by ϕ, and consider the fermionic variation (once again, we set the gauge fields to zero)

$$\delta\Psi = 2\Gamma^{k5}\varepsilon\partial_k\phi + 2\Gamma^{r5}\varepsilon\partial_r\phi - 8\alpha\phi\varepsilon\,. \tag{5.22}$$

Translating this into five-dimensional gamma matrices acting on η, one finds

$$\delta\Psi = 0 \Longleftrightarrow 2i\gamma^k\eta\partial_k\phi + 2i\gamma_r\eta\partial_r\phi + \frac{4i}{\ell}\eta\phi = 0\,. \tag{5.23}$$

Now we find that for Killing spinors satisfying $\gamma_r\eta = \eta$, the supersymmetry condition becomes $\partial_k\phi = 0$ and

$$\partial_r\phi = -\frac{2}{\ell}\phi\,, \tag{5.24}$$

or

$$\phi = \mathrm{e}^{-2r/\ell}\phi^0\,, \tag{5.25}$$

where ϕ^0 is an arbitrary constant anti-hermitean matrix in the Lie algebra of the gauge group.

Let us note the following properties of this configuration:

(1) By construction, this configuration leaves half of the supersymmetries (namely those associated with Killing spinors satisfying $\gamma_r\eta = \eta$) unbroken.

(2) It is also a solution to the equations of motion. The equation of motion is (with the mass term expressed in terms of ℓ)

$$\Box\phi = -\frac{4}{\ell^2}\phi\,. \tag{5.26}$$

On functions depending only on r, this reduces to

$$\left(\partial_r^2 + \frac{4}{\ell}\partial_r\right)\phi = -\frac{4}{\ell^2}\phi\,, \tag{5.27}$$

which is satisfied by $\phi \sim \exp(-2r/\ell)$.

(3) In the flat space limit $\ell \to \infty$, ϕ just reduces to a constant. In that limit there is a supersymmetry enhancement and ϕ^0 parametrizes the maximally supersymmetric Coulomb branch of the five-dimensional $N = 2$ theory.

For $n = 4$ the situation is quite similar. We now have two scalar fields which, with the above conventions, would most naturally be called ϕ_3 (say) and ϕ_5. But I will just call them $\phi_{1,2}$. Vanishing of the supersymmetry transformation in this case (for the $\gamma_r = +1$ Killing spinors) forces these fields to be x^k-independent and to commute, and the r-dependence is determined by

$$\partial_r\phi_{1,2} = -\frac{1}{\ell}\phi_{1,2}\,, \tag{5.28}$$

leading to

$$\phi_{1,2} = \mathrm{e}^{-r/\ell} \phi^0_{1,2} \,. \tag{5.29}$$

These are once again half-supersymmetric solutions to the equations of motion, which in this case read

$$\left(\partial_r^2 + \frac{3}{\ell}\partial_r\right)\phi_{1,2} = -\frac{2}{\ell^2}\phi_{1,2}\,, \tag{5.30}$$

and tend to the standard Coulomb branch of $N = 2$ $n = 4$ SYM as $\ell \to \infty$. Once again in that limit one finds a supersymmetry enhancement.

6. Open issues: interpretation and applications

Above we have constructed two families of curved space counterparts of the standard Poincaré supersymmetric SYM theories which are globally supersymmetric on manifolds admitting Killing spinors, and we also began a preliminary investigation of their properties. But clearly a large number of issues still remain to be understood.

(1) Foremost among them is perhaps the relevance of these theories to the dynamics of D-branes. For this one might also want to consider spacetimes of the form $M = \Sigma \times \mathbb{R}$ where Σ admits Killing spinors. The analysis closely resembles the one for euclidean theories on Σ described in section 3.4.

If these theories play a role in that context, what are the consequences of the unusual properties of the Coulomb branch we have found in section 5? Where would one expect the mass or cubic potential terms to show up in applications? What about BPS configurations with non-trivial gauge fields (monopoles) in these theories? What is the relation to the BPS configurations in AdS space studied e.g. in [6, 7]? What is the relation to the AdS calibrations of [10, 11]? Are there interesting cohomological versions of these theories?

(2) One might also want a better understanding of the superalgebras underlying these theories, depending on the number of available Killing spinors. What about the $(d = 10,\ n = 8, 9)$ theories? How is the problem to construct such theories related to the absence of conventional AdS superalgebras beyond $n = 7$? What about central charges and the addition of matter fields?

(3) It would also be desirable to have a more conceptual understanding of the existence of these two classes of theories. For the family A theories a possible approach may be the following. There is a one-to-one correspondence between (riemannian, positive) Killing spinors on M and parallel spinors on

the so-called cone CM over M [17] (see e.g. [36] for a survey of these matters in the AdS/CFT context), with similar results for other signatures and signs. Thus the parallel spinors on CM appear to play a dual role. On the one hand, they assure the supersymmetry of SYM theory on CM. On the other hand, they are invoked to establish the existence of Killing spinors on M and hence supersymmetry of SYM theory on M. It is therefore natural to wonder if these two appearances of parallel spinors are related and if, indeed, a straightforward dimensional reduction of the supersymmetric theory on CM might not have been a less roundabout way of arriving at the theory on M.

The problem with a naive dimensional reduction of a theory on CM to one on M is that there is no isometry in the cone direction but only a homothety. This suggests that perhaps one way to reduce a theory on CM to a theory on M is to perform a Scherk-Schwarz like reduction or gauging along the radial direction. The structure of the Family A theories is certainly suggestive: one "internal" gamma-matrix Γ_1 is singled out, which should be identified with Γ_r, and the mass terms could arise from a Scherk-Schwarz like reduction. However, so far I have been unable to derive these theories in this way.

(4) For the theories in family B, an altogether different idea appears to be required to account for the Chern-Simons-like terms. The appearance of such a term in the n-dimensional action suggests an $(n+3)$-dimensional origin with a true CS term living in those extra three dimensions. Thus one should have a coupling

$$\int F^{(n)}(AdA+\cdots)\,,$$

where $F^{(n)}$ is proportional to the volume form on M. Thinking of this as a RR field strength, one recognizes the Wess-Zumino coupling of a D$(n+2)$-brane world volume to a D$(n-2)$-brane via the instanton action $\mathrm{Tr}\,F\wedge F$. E.g. for $n=5$ and AdS_5 one has a D3-D7 brane system. And indeed in the near-horizon limit of such a system one obtains $AdS_5\times X_5$, where $X_5=S^5/\mathbb{Z}_2$ has a fixed S^3 over which the $D7$-branes are wrapped [37, 38] and the $F^{(5)}$ is proportional to the volume element on AdS_5 (plus its Hodge dual). Thus the D7-O7 couplings of the form

$$\int C^{(4)}\wedge \mathrm{Tr}\,F\wedge F$$

could be responsible for the Chern-Simons like terms in the five-dimensional gauge theory obtained by reduction of the worldvolume theory of the D7-branes to AdS_5.

Of course, even if one can trace the Chern-Simons terms back to these configurations (and hence the corresponding supergravity theory), one still needs to understand why they are required by supersymmetry for a gauge theory on AdS_n (or some other space-time admitting Killing spinors). However, perhaps the above considerations may at least provide a first step to such an understanding.

Alternatively, the existence of such terms in the action could be deduced from considerations as in [29], where D-brane actions in non-trivial antisymmetric tensor field backgrounds (and hence also non-trivial curvature by the Einstein equations) are studied.

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A. Some useful identities for fermion bilinears

To understand the hermiticity properties of fermionic mass terms, which play an important role in the discussion of section 3, and in order to facilitate other manipulations, it is useful to know some identities for spinor bilinears involving gamma-matrices. First of all, let us introduce the unitary matrices $A_\pm, B_\pm, C_\pm$ by

$$\begin{aligned} \Gamma_M^\dagger &= \pm A_\pm \Gamma_M A_\pm^{-1} \,, \\ \Gamma_M^* &= \pm B_\pm \Gamma_M B_\pm^{-1} \,, \\ \Gamma_M^T &= \pm C_\pm \Gamma_M C_\pm^{-1} \,. \end{aligned} \tag{A.1}$$

We can always choose $A_- = \Gamma_0 = -A_-^\dagger$, and for d even for A, B and C the $\pm$ matrices are related by multiplication by $\Gamma^{(d+1)}$. For a general analysis see e.g. [39].

Majorana spinors are characterized by the condition

$$\Psi^* = B_\pm \Psi \,, \tag{A.2}$$

which is consistent provided that

$$B_\pm^* B_\pm = \mathbb{I} \,. \tag{A.3}$$

Then for a Majorana spinor one has

$$\bar{\Psi} = \Psi^\dagger A_- = \Psi^T B_\pm^T A_- \,. \tag{A.4}$$

But one can easily check that, given the properties of A and B, one has

$$B_{\pm}^{T}A_{-}\Gamma_{M}(B_{\pm}^{T}A_{-})^{-1}=\mp\Gamma_{M}^{T}\,, \tag{A.5}$$

and thus one can identify

$$C_{\mp}=B_{\pm}^{T}A_{-}\,. \tag{A.6}$$

Hence the Majorana condition can also be written as

$$\bar{\Psi}=\Psi^{T}C_{\mp}\,, \tag{A.7}$$

which is perhaps more familiar. For the Majorana(-Weyl) theories in $d=3+1$ and $d=9+1$, we will usually choose $B=B_{+}$ to obtain

$$B=B_{+}\Longrightarrow\bar{\Psi}=\Psi^{T}C_{-}\,. \tag{A.8}$$

In a Majorana basis of real gamma-matrices, one can always choose $B_{+}=\mathbb{I}$ and $A_{-}=C_{-}$, since $\Gamma_{M}^{\dagger}=\Gamma_{M}^{T}$ and hence Majorana spinors are real in such a basis.

Now let us look quite generally at a spinor bilinear

$$\bar{\Psi}\Gamma^{[p]}\Phi\,. \tag{A.9}$$

If Ψ and Φ are chiral spinors, then it is easy to see that this bilinear is zero if p is even and Ψ and Φ have the same chirality (and likewise is zero if p is odd and Ψ and Φ have opposite chiralities). To see this one can calculate, using $\Gamma^{(d+1)\dagger}=\Gamma^{(d+1)-1}=\Gamma^{(d+1)}$,

$$\overline{\Gamma^{(d+1)}\Psi}\Gamma^{[p]}\Phi=(-1)^{p+1}\bar{\Psi}\Gamma^{[p]}\Gamma^{(d+1)}\Phi\,, \tag{A.10}$$

from which the claim follows. Now let us check under which conditions the corresponding mass term is hermitean. To that end we calculate, noting an extra minus sign due to working with anticommuting spinors,

$$(\bar{\Psi}\Gamma^{[p]}\Phi)^{\dagger}=\eta_{p}\bar{\Phi}\Gamma^{[p]}\Psi \tag{A.11}$$

(η_p was defined in (2.25)) so that $\bar{\Psi}\Gamma^{[p]}\Psi$ is hermitean for $\eta_p=+1$, i.e. $p=0,3,4,7,8,$ ... while for $\eta_p=-1$, one has to multiply this term by i to obtain a hermitean mass term.

For Ψ and Φ Majorana, one has, using also $C^{T}=-C$ (in a Majorana basis)

$$\bar{\Psi}\Gamma^{[p]}\Phi=(\bar{\Psi}\Gamma^{[p]}\Phi)^{T}=\eta_{p}\bar{\Phi}\Gamma^{[p]}\Psi \tag{A.12}$$

consistent with the fact that in a Majorana basis transposition and hermitean conjugation are the same operation. Thus the potential mass term $\bar{\Psi}\Gamma^{[p]}\Psi$ is zero unless $\eta_p=+1$ (and in this case we are not permitted to render the mass term hermitean for $\eta_p=-1$ by multiplying it by i).

JHEP11(2000)023

Summarizing the above discussion, we see that for the $d = 2+1$ Majorana theory, the only posibility is $p = 3$, equivalent to $p = 0$ because Γ_{012} is a multiple of the identity in that case. Likewise, for the $d = 3+1$ Majorana theory, the only possibilities are $p = 0, 3, 4$. For the chiral version of that theory, we have $p = 1$ or $p = 3$ (with imaginary and real coefficients respectively, related to the fact that $\Gamma^{(5)}$ has a factor of i). For the chiral theory in $d = 5+1$, one necessarily has p odd, and therefore either $p = 1$ (equivalent to $p = 5$) with a factor of i, or $p = 3$ with a real coefficient. We will find supersymmetric gauge theories for either choice of mass term. Finally, the only possibility for the Majorana-Weyl theory in $d = 9+1$ is $p = 3$.

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